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Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 427–437

TIME-DEPENDENT PROBLEMS OF THE CONCENTRATION OF ELASTIC STRESSES NEAR A CONICAL DEFECT†

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(*Received* 25 *March* 2004)

Problems of the stress state of an unbounded elastic medium containing a conical defect (a crack or a fine inclusion) and loaded with an unsteady torsional elastic wave or an impact load in the form of a torsional moment are solved. Using the discontinuous solutions of the equations of dynamic elasticity for a conical defect constructed earlier, the problem is reduced to an integrodifferential equation (in the case of a crack) or an integral equation (in the case of an inclusion) in the space of Laplace transforms. A method is proposed for solving these last equations based on the combined use of the method of orthogonal polynomials and the discretization of the equation with respect to time. A formula is obtained and the time-dependence of the stress intensity factor near the edge of the crack and the angle of rotation of the inclusion is calculated.

Problems concerning the stress state of an elastic medium containing a conical defect have been solved in a static formulation [1] using discontinuous solutions of Lame's equation in the Gutman form [2]. An axisymmetric problem in the theory of elasticity for a space with a conical cut was considered in [3].

Below, we propose an efficient method for solving a new dynamic time-dependent problem of the stress state of an elastic medium when there is a conical defect in it either in the form of a fine inclusion (a conical shell) or in the form of a cut along a conical surface. In order to check the effectiveness of the method, mechanical characteristics are calculated for actual materials and dimensions of the defect. A solution is constructed using the method of discontinuous solutions, which enables the problem to be reduced to a system of one-dimensional integrodifferential or integral equations in the space of Laplace transforms with respect to time.

The method of discontinuous solutions is an original special implementation of the method of potentials, adapted for the efficient use of integral transforms and which is related each time to a suitable orthogonal system of coordinates (which is dictated by the formulation of the problem). Also, whereas in the general scheme in the method of potentials a discontinuous solution is constructed in the form of a linear combination of the potentials of a single and double layer (for which a complex preliminary procedure for elucidating the mechanical meaning of the above-mentioned potentials has to be carried out), in the method of discontinuous solutions the discontinuities in the required mechanical quantities are specified or, more accurately, are assumed to be temporarily known. Here, some of the discontinuities are, in fact, known from the formulation of the problem, and the rest are found by satisfying the conditions on the defect. Subsequent application of an inverse Laplace transformation, taking account of the contraction theorem, reduces the equations to two-dimensional equations. The solution of the equations is based on the combined use of time-discretization and the method of orthogonal polynomials, which leads to the solution of a sequence of infinite systems of linear algebraic equations.

†*Prikl. Mat. Mekh.* Vol. 69, No. 3, pp. 469–480, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.05.010

1. FORMULATION OF THE PROBLEM

A conical defect is contained in an unbounded elastic medium and the surface of this defect is described in a spherical system of coordinates by the relations

$$a \le r \le b, \quad \theta = \omega, \quad -\pi \le \varphi \le \pi \tag{1.1}$$

On passing through the surface (1.1), the displacements $u_\varphi(r, \theta, t) \equiv u(r, \theta, t)$ and the stresses $\tau_{\theta\varphi}(r, \theta, t) \equiv \tau(r, \theta, t)$ undergo discontinuities of the first kind with jumps

$$\begin{aligned} \langle u(r, \omega, t)\rangle &= u(r, \omega - 0, t) - u(r, \omega + 0, t) \\ \langle \tau(r, \omega, t)\rangle &= \tau(r, \omega - 0, t) - \tau(r, \omega + 0, t) \end{aligned} \tag{1.2}$$

When the defect (1.1) is a crack, the displacements undergo discontinuities but, if the defect is a thin, absolutely rigid, conical shell which is coupled to the medium, only the stresses are discontinuous. It is assumed that an elastic unsteady torsional wave which travels from infinity [4]

$$\overset{0}{u}(r, \theta, t) = \sin\theta[H(ct - r\cos\theta) - H(ct - r\cos\theta - ct_0)] \tag{1.3}$$

interacts with the crack or with the fixed shell, where $H(z)$ is the Heaviside unit function, t_0 is the duration of the pulse and c is the wave propagation velocity.

The displacement $u(r, \theta, t)$ satisfies the equation [5]

$$(r^2u')' + \frac{(\sin\theta u^{\cdot})^{\cdot}}{\sin\theta} - \frac{u}{\sin^2\theta} = \frac{r^2}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{1.4}$$

where a prime indicates a derivative with respect to the variable r and a dot denotes a derivative with respect to the variable θ.

In the case of a crack, the condition that there are no stresses on the edges ($\theta = \omega - 0$ and $\theta = \omega + 0$) of the crack

$$\tau(r, \theta, t)|_{\theta = \omega} = 0 \tag{1.5}$$

has to be satisfied. In the case of zero initial conditions, it is necessary to determine the stress intensity factor at the edges of the crack.

If the shell is fixed, it is assumed that a dynamic load acts in the form of the elastic torsional wave (1.3) which is incident from infinity. The condition for the continuity of the displacements

$$u(r, \theta, t)|_{\theta = \omega} = 0 \tag{1.6}$$

is then satisfied on the surface of the inclusion. The reactive moment in the shell has to be determined in the case of zero initial conditions.

If, however, the shell is not fixed, then it is assumed that it is loaded with an impact load in the form of a torque $M = AH(t)$ under the action of which it may be rotated around an axis of revolution by an angle $\alpha(t)$ together with the elastic medium. In this case, the condition on the surface of the inclusion takes the form

$$u(r, \theta, t)|_{\theta = \omega} = \alpha(t)r \tag{1.7}$$

and it is necessary to determine the unknown angle of rotation of the inclusion $\alpha(t)$.

2. REDUCTION OF THE PROBLEM TO AN INTEGRODIFFERENTIAL EQUATION. THE CASE OF A CRACK

We will represent the displacement and stress fields in the form

$$\begin{Vmatrix} u_p(r, \theta) \\ \tau_p(r, \theta) \end{Vmatrix} = \begin{Vmatrix} \overset{0}{u}_p(r, \theta) + \overset{1}{u}_p(r, \theta) \\ \overset{0}{\tau}_p(r, \theta) + \overset{1}{\tau}_p(r, \theta) \end{Vmatrix} \tag{2.1}$$

where p is the parameter of the Laplace transformation with respect to time, u_p^0 and τ_p^0 are the Laplace transforms of the displacements and stresses caused by the incident wave when there is no defect in the elastic medium, and u_p^1 and τ_p^1 are the transforms of the required perturbed displacement and stress fields due to the existence of the defect (1.1) in the medium. The perturbed field is constructed in the form of a discontinuous solution [6] of the equation of motion for the defect (1.1), in which the discontinuity of the displacements and stresses is contained.

The Laplace transform of the discontinuous solution of the torsional equation (1.4) has the form [7]

$$u_p^1(r,\theta) = s\sin\omega\frac{\partial}{\partial\theta}\left[\int_0^\infty \langle\Psi_p(\rho,\omega)\rangle\frac{\partial}{\partial\omega}G_0(rs,\rho s;\theta,\omega) - \right.$$
$$\left. - \langle\Psi_p^{\cdot}(\rho,\omega)\rangle G_0(rs,\rho s;\theta,\omega)d\rho\right] \tag{2.2}$$

$$G_0(rs,\rho s;\theta,\omega) = \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)P_k(\cos\omega)P_k(\cos\theta)J_k(rs,\rho s) \tag{2.3}$$

$$J_k(rs,\rho s) = \frac{1}{s\sqrt{r\rho}}\begin{cases} I_\nu(rs)K_\nu(\rho s), & r<\rho \\ I_\nu(\rho s)K_\nu(rs), & r>\rho\end{cases};\quad \nu = k+\frac{1}{2},\quad s = \frac{p}{c} \tag{2.4}$$

Here, I_ν and K_ν are modified Bessel functions of half-integral order, P_k are Legendre functions and Ψ_p is the Laplace transform of the wave function.

It is required to relate the discontinuity of the function Ψ_p and the discontinuity of its normal derivative, which occurs in the discontinuous solution (2.2), to the discontinuities of the mechanical quantities. We shall now deal with this, taking account of the formula which relates the transforms of these discontinuities and the transforms of the discontinuities of the displacements and stresses (G is Young's modulus)

$$\langle\Psi_p(r,\omega)\rangle = -s\int_0^\infty[sG^{-1}\langle\tau_p^1(\rho,\omega)\rangle + 2\operatorname{ctg}\omega\langle u_p^1(\rho,\omega)\rangle]J_0(rs,\rho s)d\rho \tag{2.5}$$

$$\langle\Psi_p^{\cdot}(r,\omega)\rangle = -\langle u_p^1(r,\omega)\rangle \tag{2.6}$$

Taking boundary condition (1.5) into account, in accordance with relation (2.5), we write the relation for determining the discontinuity of the wave function in the form

$$\langle\Psi_p(r,\omega)\rangle = -2s\operatorname{ctg}\omega\int_0^\infty\langle u_p^1(\rho,\omega)\rangle J_0(rs,\rho s)d\rho$$

The transforms of the discontinuities of the wave function and its derivative will be found if the transform of the discontinuity of the displacements $\langle u_p^1(r,\omega)\rangle$ is known. With the aim of determining this, we write the condition on the crack (1.5) which, according to the representation (2.1), takes the form

$$\tau_p^1(r,\theta)|_{\theta=\omega-0} = -\tau_p^0(r,\omega) \tag{2.7}$$

The relation

$$rG^{-1}\tau_p^1(r,\omega-0) = 2\operatorname{ctg}\omega\Psi_p^{\cdot}(r,\omega-0) + L_s\Psi_p(r,\omega-0) \tag{2.8}$$

is taken into account for determining the left-hand side of equality (2.7), where

$$L_sf = (r^2f')' - (rs)^2f \tag{2.9}$$

$$\Psi_p(r, \omega - 0) = \frac{1}{2}\langle\Psi_p(r, \omega)\rangle - \sin\omega\int_0^\infty\Big[\langle\Psi_p(\rho, \omega)\rangle\frac{\partial}{\partial\omega}G_0(rs, \rho s; \theta, \omega)|_{\theta=\omega} -$$

$$- \langle\Psi_p^{\bullet}(\rho, \omega)\rangle G_0(rs, \rho s; \theta, \omega)|_{\theta=\omega}\Big]d\rho \tag{2.10}$$

$$\Psi_p^{\bullet}(r, \omega - 0) = \frac{\partial}{\partial\theta}\Psi_p(r, \omega - 0)$$

Into the second relation of (2.10), we now substitute the values of the transforms of the discontinuities of the wave function and its derivative, expressed in terms of the transform of the discontinuity of the displacements which, in turn, we substitute into relation (2.8). It is therefore necessary to solve the integrodifferential equation in order to determine the discontinuity of the displacements.

We now make a replacement in the resulting formulae. We will temporarily assume that the Laplace transformation parameter s is positive and put

$$r = xs^{-1}, \quad \rho = \xi s^{-1} \tag{2.11}$$

We now carry out the replacement (2.11) in relation (2.9) and in the second relation of (2.10) and introduce the notation

$$X(r, \omega) = \langle u_p^1(r, \omega)\rangle, \quad L_x = x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} + x^2 \tag{2.12}$$

Taking into account the replacement (2.11), we write relation (2.8) in the form

$$xs^{-1}G^{-1}\tau_p^1(xs^{-1}, \omega - 0) = -2s^{-1}\frac{\cos^2\omega}{\sin\omega}L_x\int_0^\infty X(\xi s^{-1}, \omega)S_3(x, \xi; \omega)d\xi +$$

$$+ \cos\omega L_x\int_0^\infty X(\xi s^{-1}, \omega)S_2(x, \xi; \omega)d\xi - s^{-1}\cos\omega L_x\int_0^\infty X(\xi s^{-1}, \omega)S_1(x, \xi; \omega)d\xi - \tag{2.13}$$

$$- \sin\omega L_x\int_0^\infty X(\xi s^{-1}, \omega)S_0(x, \xi; \omega)d\xi$$

where

$$S_3(x, \xi; \omega) = \sum_{k=1}^\infty \frac{2k+1}{[k(k+1)]^2}[P_k^1(\cos\omega)]^2 J_k(x, \xi)$$

$$S_2(x, \xi; \omega) = \sum_{k=1}^\infty \frac{2k+1}{[k(k+1)]}P_k^1(\cos\omega)P_k(\cos\omega)J_k(x, \xi)$$

$$S_1(x, \xi; \omega) = \sum_{k=1}^\infty \frac{2k+1}{[k(k+1)]}P_k^1(\cos\omega)P_k(\cos\omega)(J_0(x, \xi) - J_k(x, \xi))$$

$$S_0(x, \xi; \omega) = \sum_{k=1}^\infty (2k+1)[P_k(\cos\omega)]^2 J_k(x, \xi)$$

The relation $L_xJ_k(x, \xi) = k(k+1)J_k(x, \xi)$ has been used here.

In order to separate out the singular part from the kernels of the resulting integral operators, we consider the asymptotic form of the expressions obtained when $k \to \infty$. According to the formulae which define the behaviour of Bessel functions for large values of the order [8, formulae (9.7.7) and (9.7.8)], we obtain that

$$J_k(x, \xi) \approx \frac{1}{2k+1}\left(\frac{x}{\xi}\right)^k$$

The asymptotic behaviour of Legendre functions when $k \to \infty$ [9] gives

$$[P_k(\cos\omega)]^2 = Q_k(\omega) + O(k^{-3/2}), \quad Q_k(\omega) = \frac{1+\sin(2k+1)\omega}{\pi k \sin\omega}$$

Substituting the resulting expressions into the series $S_0(x, \xi; \omega)$, we find that the asymptotic behaviour of the kth term of the series, which is generated by the last term on the right-hand side of equality (2.13), is described by the expression $Q_k(\omega)(x/\xi)^k$.

The following operation is carried out in order to separate out the singularity in the last term of relation (2.13): the series $S_0(x, \xi; \omega)$ is represented in the form of the sum of two terms

$$S_0(x, \xi; \omega) = \left(\sum_{k=1}^{N} + \sum_{k=N+1}^{\infty}\right)(2k+1)[P_k(\cos\omega)]^2 J_k(x, \xi)d\xi$$

The terms of the series for large value of k are replaced by their asymptotic expressions. In the resulting relation, the finite sum $\sum_{k=1}^{N} Q_k(\omega)(x/\xi)^k$ is then added and subtracted, which reduces the series $S_0(x, \xi; \omega)$ to the form

$$S_0(x, \xi; \omega) = -\frac{1}{\pi\sin\omega}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x}{\xi}\right)^k + \alpha^1(x, \xi)$$

$$\alpha^1(x, \xi) = \sum_{k=1}^{\infty}\tilde{Q}_k(\omega)(x/\xi)^k + \sum_{k=1}^{N}(2k+1)[P_k(\cos\omega)]^2 J_k(x, \xi) - \sum_{k=1}^{N} Q_k(\omega)(x/\xi)^k$$

$$\tilde{Q}_k(\omega) = \frac{\sin(2k+1)\omega}{\pi k\sin\omega}$$

The series in the expressions for $S_0(x, \xi; \omega)$ are summed using well-known formulae [10], formulae (5.4.9) (12, 13) and (5.2.4) (4)]. Finally, we obtain the last term on the right-hand side of equality (2.13) in the form

$$\frac{1}{\pi}L_x\int_0^{\infty} X(\xi s^{-1}, \omega)\left[\ln\frac{1}{|\xi - x|} + \alpha^1(x, \xi) + \alpha^2(x, \xi)\right]d\xi, \quad \alpha^2(x, \xi) = \begin{cases} \dfrac{\ln\xi}{s\xi}, & x < \xi \\ \dfrac{\ln x}{sx} + \dfrac{\xi - x}{s(x\xi)^{3/2}}, & x > \xi \end{cases}$$

Using the formulae for the asymptotic behaviour of Bessel and Legendre functions for a large value of the order, it can be shown that the first three terms on the right-hand side of equality (2.13) make no contribution to the singular part.

We now return to the initial variables in equality (2.13) and separate out the term carrying the greatest singularity. The second derivative is such a term. We then satisfy the condition on the crack (2.7). As a result, we obtain

$$\frac{d^2}{dr^2}\int_a^b X(\rho, \omega)\ln\frac{1}{|r-\rho|}d\rho + \int_a^b X(\rho, \omega)R(r, \rho, s)d\rho = f(r, s) \tag{2.14}$$

where

$$R(rs, \rho s) = \frac{2}{rs}\frac{d}{dr}[-\ln|r-\rho| + \alpha^1(rs, \rho s) + \alpha^2(rs, \rho s)] +$$

$$+ \operatorname{ctg}\omega\frac{d^2}{dr^2}[s^{-1}S_1(rs, \rho s; \omega) + S_2(rs, \rho s; \omega) + 4\operatorname{ctg}\omega s^{-1}S_3(rs, \rho s; \omega)] +$$

$$+ \operatorname{ctg}\omega\frac{1}{r}\frac{d}{dr}[4s^{-1}S_1(rs, \rho s; \omega) + 4S_2(rs, \rho s; \omega) + 8\operatorname{ctg}\omega s^{-1}S_3(rs, \rho s; \omega)] -$$

$$- \operatorname{ctg}\omega[sS_1(rs, \rho s; \omega) - 2s^2S_2(rs, \rho s; \omega) - 4\operatorname{ctg}\omega S_3(rs, \rho s; \omega)]$$

$$f(r, s) = -\frac{Gs^{-1}r^{-3}}{\sin\omega}\tau_p^0(r, \omega)$$

3. REDUCTION OF THE PROBLEM TO AN INTEGRAL EQUATION. THE CASE OF AN INCLUSION

Taking representation (2.1) into account, we write the condition on the inclusion (1.6) in the form

$$u_p^1(r, \theta)\big|_{\theta = \omega - 0} = -u_p^0(r, \omega) \tag{3.1}$$

The left-hand side of equality (3.1) is determined by formula (2.2), taking account of the fact that the shell is coupled to an elastic medium, which means that, in relations (2.5) and (2.6), it is necessary to put

$$\langle u_p^1(r, \omega)\rangle = 0 \tag{3.2}$$

The required transform of the discontinuity of the stresses is denoted by $\Phi_p(r) = \langle\tau_p^1(r, \omega)\rangle$. Taking (2.5), (2.6) and (3.2) into account, we then write relation (2.2) in the form

$$u_p^1(r, \theta) = -s^2\frac{\sin\omega}{G}\frac{\partial}{\partial\theta}\int_0^\infty \xi\Phi_p(\xi)\int_0^\infty \frac{\partial}{\partial\omega}G_0(rs, \rho s; \theta, \omega)J_0(\rho s, \xi s)d\xi d\rho \tag{3.3}$$

Taking the limit $\theta = \omega - 0$ in equality (3.3) and using condition (3.1), we obtain the following integral equation in the space of Laplace transforms

$$\begin{gathered}\int_a^b \sigma\Phi_p(\sigma)K_p(r, \sigma)d\sigma = -Gu_p^0(r, \omega) \\ K_p(r, \sigma) = \sin\omega\sum_{k=1}^\infty \frac{2k+1}{s^2[k(k+1)]^2}\left|P_k^1(\cos\omega)\right|^2(J_0(rs, \sigma s) - J_k(rs, \sigma s))\end{gathered} \tag{3.4}$$

Investigating the asymptotic form of the kernel when $k \to \infty$ and carrying out operations with respect to the summation of the weakly converging part of the series in relation (3.4) in a similar manner to the procedure carried out above in the case of Eq. (2.9), we obtain the integral equation

$$\frac{\sin\omega}{\pi}\int_a^b \ln\frac{1}{|r-\sigma|}\Phi_p(\sigma)d\sigma + \frac{\sin\omega}{\pi s^2}\int_a^b M(r, \sigma, s)\Phi_p(\sigma)d\sigma = -Gu_p^0(r)$$

$$M(r, \sigma, s) = \sum_{k=1}^{\infty} \frac{\sin(2k+1)\omega}{k}\left(\frac{r}{\sigma}\right)^k - \sum_{k=1}^{N} \frac{2k+1}{k(k+1)}[P_k^1(\cos\omega)]^2 J_k(rs, \sigma s) + \tag{3.5}$$

$$+\frac{1}{s^2}\left(\frac{r}{\sigma}+1\right)\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\left(\frac{r}{\sigma}\right)^k + \begin{cases} \dfrac{\ln\sigma}{s\sigma}, & r<\sigma \\ \dfrac{\ln r}{sr} + \dfrac{\sigma - r}{sr\sigma}, & r>\sigma \end{cases}$$

4. SOLUTION OF THE CRACK PROBLEM

In order to apply the contraction theorem for an integral Laplace transform to equality (2.14), the functions

$$\Phi(\rho, \omega, t) = L^{-1}(X(\rho, \omega)), \quad D(r, \rho, t) = L^{-1}(R(rs, \rho s)), \quad F(r, t) = L^{-1}(f(rs))$$

are introduced. Here, L^{-1} is the inverse Laplace transformation operator. Tabulated inversion formulae [11]

$$L^{-1}[I_\vartheta(ap)K_\vartheta(bp)] = Z_\vartheta(t, a, b) = \begin{cases} 0, & t<a-b \\ \dfrac{1}{2\sqrt{ab}}P_{\vartheta-1/2}((2ab)^{-1}(a^2+b^2-t^2)), & a-b<t<a+b \\ \dfrac{1}{\pi\sqrt{ab}}Q_{\vartheta-1/2}((2ab)^{-1}(t^2-a^2-b^2)), & t>a+b \end{cases}$$

are used to invert the function $R(rs, \rho s)$.

As a result, we write the original function $R(rs, \rho s)$ in the form

$$D(r, \rho, t) = \operatorname{ctg}\omega \sum_{k=1}^{\infty} (2k+1)[k(k+1)]^{-1} P_k^1(\cos\omega)P_k(\cos\omega)\Big[{}_1\Omega_k'(t, r, \rho) +$$

$$+\frac{4}{r}{}_1\Omega_k(t, r, \rho) - Z_0(t, r, \rho) + Z_\vartheta(t, r, \rho) +$$

$$+{}_2\Omega_k''(t, r, \rho) + \frac{1}{r}{}_2\Omega_k'(t, r, \rho) + 2\left[\frac{\partial Z_\vartheta(t, r, \rho)}{\partial t} + Z_\vartheta(0, r, \rho)\right] +$$

$$+4\operatorname{ctg}^2\omega \sum_{k=1}^{\infty}(2k+1)[k(k+1)]^{-2}[P_k^1(\cos\omega)]^2\Big[{}_3\Omega_k'(t, r, \rho) + \frac{2}{r}{}_3\Omega_k(t, r, \rho) +$$

$$+{}_1\Omega_k'(t, r, \rho) + Z_\vartheta(t, r, \rho)\Big] + \frac{2}{r|r-\rho|} + \frac{2}{r}\sum_{k=1}^{N}(2k+1)[P_k(\cos\omega)]^2 {}_2\Omega_k(t, r, \rho) + \alpha^2(t, r, \rho)$$

$${}_1\Omega_k(t, r, \rho) = \int_0^1 \tilde{t}[Z_0'(t-\tilde{t}, r, \rho) - Z_\vartheta'(t-\tilde{t}, r, \rho)]d\tilde{t}$$

$${}_2\Omega_k(t, r, \rho) = \int_0^1 Z_\vartheta(t-\tilde{t}, r, \rho)d\tilde{t}, \quad {}_3\Omega_k(t, r, \rho) = \int_0^t \tilde{t}Z_\vartheta'(t-\tilde{t}, r, \rho)d\tilde{t}$$

Application of an inverse Laplace transformation to Eq. (2.14) using the contraction theorem leads to an integrodifferential equation of the form

$$\frac{d^2}{dr^2}\int_a^b \ln\frac{1}{|r-\rho|}\Phi(t, \rho, \omega)d\rho + \int_0^t\int_a^b D(t-\tilde{t}, r, \rho)\Phi(\tilde{t}, \rho, \omega)d\rho d\tilde{t} = F(t, r) \tag{4.1}$$

Its approximate solution is constructed by the combined use of discretization of the equation with respect to time and the method of orthogonal polynomials. The time interval $[0, T]$, during which the interaction of the elastic wave with the defect is investigated, is subdivided with a step size $h = T/N$ into intervals $[\tilde{t}_k, \tilde{t}_{k+1}]$, where $\tilde{t}_k = kT/N$ $(k = 1, 2, \ldots, N)$, and the integral with respect to the time $\tilde{t}$ is replaced in Eq. (4.1) by Simpson's quadrature formula, the nodes of which are denoted by A_k. We have

$$\frac{d^2}{dr^2}\int_a^b \ln\frac{1}{|r-\rho|}\Phi_n(\rho)d\rho + \sum_{k=1}^{N} A_k\int_a^b D(t_n - \tilde{t}_k, r, \rho)\Phi_k(\rho)d\rho = F_n(r) \tag{4.2}$$

where

$$n = 1, 2, \ldots, N, \quad \Phi_n(\rho) = \Phi(t_n, \rho), \quad F_n(r) = F(t_n, r)$$

Before using the method of orthogonal polynomials, which is based on the application of the spectral relation for Chebyshev polynomials of the second kind [6]

$$\frac{d^2}{dx^2}\frac{1}{\pi}\int_{-1}^{1} \ln\frac{1}{|x-y|}\sqrt{1-y^2}U_n(y)dy = -(n+1)U_n(x), \quad |x| \le 1, \quad n = 1, 2, \ldots \tag{4.3}$$

we make the change of variables in Eq. (4.2)

$$r = \delta_+ + \delta_- x, \quad \rho = \delta_+ + \delta_- y; \quad \delta_+ = \frac{a+b}{2}, \quad \delta_- = \frac{b-a}{2} \tag{4.4}$$

We construct the solution of the system of equations obtained as a result of this change of variables in the form

$$\Phi_n(\delta_+ + \delta_- y) = \sqrt{1-y^2}\sum_{l=0}^{\infty} \Phi_l^{(n)} U_l(y) \tag{4.5}$$

After applying the standard scheme in the method of orthogonal polynomials [6], we arrive at sequences $(n = 1, 2, \ldots, N)$ of infinite systems of linear algebraic equations

$$Y_l \Phi_l^{(n)} - \sum_{k=1}^{N} A_k \sum_{m=0}^{\infty} B_{kml}\Phi_m^{(k)} = F_l^{(n)}, \quad l = 0, 1, 2, \ldots; \quad n = 1, 2, \ldots, N \tag{4.6}$$

where

$$Y_l = \frac{\delta_-^2}{\pi(l+1)}, \quad B_{kml} = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1-x^2}\sqrt{1-y^2}U_m(y)U_l(x)D(t_n - \tilde{t}_k, x, y)dxdy$$

$$F_l^{(n)} = \int_{-1}^{1} \sqrt{1-x^2}U_l(x)F_n(x)dx$$

Hence, an infinite system of linear algebraic equations

$$Y_l \Phi_l^{(i)} - \sum_{m=0}^{\infty} B_{ml}\Phi_m^{(i)} = F_l^i - \sum_{k=1}^{i-1} A_k \sum_{m=0}^{\infty} B_{km}\Phi_m^{(k)}, \quad l = 0, 1, 2, \ldots; \quad i = 2, 3, \ldots, N \tag{4.7}$$

is obtained for determining each actual value of $\Phi_l^i = \Phi_l$ $(l = 0, 1, 2, \ldots)$.

These systems can be solved approximately by the reduction method. Its applicability is proved using a scheme, described earlier in [6], which is based on the proof of the convergence of the series

$$G_1 = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} |B_{ml}|^2, \quad G_2 = \sum_{l=0}^{\infty} |F_l|^2$$

To calculate the stress intensity factor (SIF) at the edges of the crack

$$N^{+} = \lim_{r \to a-0} [\sqrt{a-r}\tau(r, \omega, t)] \tag{4.8}$$

we make the replacement (4.4) and, after discretization in time, we obtain

$$N^{+}(t_n) = \frac{\sqrt{b-a}}{\sqrt{2}} \lim_{x \to -1-0} [\sqrt{-x-1}\tau(\delta_{+} + \delta_{-}x, \omega, t_n)], \quad n = 1, 2, \dots, N \tag{4.9}$$

where

$$\begin{aligned} \tau(\delta_{+} + \delta_{-}x, \omega, t_n) = {} & \frac{2\sin\omega}{b-a}\frac{d^2}{dx^2}\int_{-1}^{1} \ln\frac{1}{|x-y|}\Phi_n(\delta_{+} + \delta_{-}y)dy + \\ & + \frac{b-a}{2}\sin\omega \sum_{k=1}^{N} A_k \int_{-1}^{1} D(t_n - \tilde{t}_k, \delta_{+} + \delta_{-}x, \delta_{+} + \delta_{-}y)\Phi_k(\delta_{+} + \delta_{-}y)dy - F_n(\delta_{+} + \delta_{-}x) \end{aligned} \tag{4.10}$$

By virtue of continuity, the last two terms make no contribution to the value of the SIF.

In order to calculate the limiting expression (4.10), it is necessary to continue the spectral relation (4.3) into the interval $|x| > 1$. To do this, a result [6] is used, according to which

$$\begin{aligned} & -\frac{d^2}{dx^2}\frac{1}{\pi}\int_{-1}^{1} \ln\frac{1}{|x-s|}\sqrt{1-s^2}U_m(s)ds = \frac{(m+1)^2 2^{m+2}}{(x-1)^{m+2}}\left[F\left(\frac{3}{2}+m, m+2; \frac{3}{2}; \frac{x+1}{x-1}\right) - \right. \\ & \left. -\frac{m+1}{2}\sqrt{\frac{1-x}{-1-x}}F\left(\frac{3}{2}+m, m+1; \frac{1}{2}; \frac{x+1}{x-1}\right)\right], \quad x < -1 \end{aligned} \tag{4.11}$$

where F is the Gaussian hypergeometric function. Using formulae (4.11) and (4.5), we take the limit in expression (4.9).

The final expression for the SIF takes the form

$$N^{+}(t_n) = \frac{\sqrt{b-a}\pi}{2}\sum_{l=0}^{\infty} (-1)^{l+1}(l+1)\Phi_l^{(n)} \tag{4.12}$$

5. SOLUTION OF THE PROBLEM FOR AN INCLUSION

An inverse Laplace transformation taking account of the contraction theorem, followed by discretization in time, according to the scheme described above, is used to solve integral equation (3.5). As a result, we obtain the system of integral equations

$$\int_a^b \ln\frac{1}{|r-\rho|}\Phi_n(\rho)d\rho + \int_a^b \rho \sum_{k=1}^{N} A_k\Phi_k(\rho)M(t_n - \tilde{t}_k, r, \rho)d\rho = -\frac{\pi G^2}{\sin\omega}u_n^0(r) \tag{5.1}$$

where

$$\Phi_n(\rho) = \Phi(t_n, \rho), \quad n = 1, 2, \dots, N$$

The solution of each integral equation of the system is based on the application of the spectral relation for Chebyshev polynomials of the first kind [6]

$$\frac{1}{\pi}\int_{-1}^{1} \ln\frac{1}{|x-y|}(1-y^2)^{-1/2}T_n(y)dy = \begin{cases} \ln 2, & n = 0 \\ n^{-1}T_n(x), & n = 1, 2, \dots \end{cases}, \quad |x| \le 1 \tag{5.2}$$

The change of variables (4.4) is made according to the scheme described above and the solution of the equation is then sought in the form of the series

$$\Phi_n(y) = (1-y^2)^{-1/2}\sum_{l=0}^{\infty}\Phi_l^{(n)}T_n(y) \tag{5.3}$$

Implementation of the standard scheme in the method of orthogonal polynomials leads to sequences of infinite systems of linear algebraic equations of the form of (4.6), the coefficients of which are

$$\begin{aligned} &Y_l = \frac{\pi}{2l}, \quad B_{kml} = \int_{-1}^{1}\int_{-1}^{1}\frac{\delta_-(\delta_+ + \delta_- y)}{\sqrt{1-x^2}\sqrt{1-y^2}}M(t_n - \tilde{t}_k, \delta_+ + \delta_- x, \delta_+ + \delta_- y)T_l(x)T_m(y)dxdy \\ &F_l^n = -\int_{-1}^{1}\frac{u_n^0(x)}{\sqrt{1-x^2}}T_l(x)dx \end{aligned} \tag{5.4}$$

Each of the infinite systems of linear algebraic equations is approximately solved by the reduction method, the applicability of which is proved using the scheme described earlier in [6].

The time-dependence of the reactive torque

$$M_0(t) = \int_a^b \tau(r, \omega, t)dr$$

was investigated in the case of a fixed inclusion. In the case of an unfixed inclusion, d'Alembert's principle is used to determine the unknown angle of rotation. According to this principle,

$$M_0(t) + AH(t) = Jd^2\alpha(t)/dt^2 \tag{5.5}$$

where $AH(t)$ is the applied torque and J is the specified moment of inertia of the inclusion about the axis of revolution.

Transferring in relation (5.5) into Laplace transform space

$$\alpha_p = M_{0p}/(Js^2) + A/(Js^3) \tag{5.6}$$

and substituting the resulting expression into condition (1.7), which has also to be written in Laplace transform space, we obtain a system of integral equations of the form (5.1), from which the coefficients of expansion (5.3) are determined. On inverting the transform (5.6) after this, we obtain the following formula for determining the angle of rotation of the inclusion

$$\alpha(t) = \frac{1}{J}\int_0^t M_0(t-\tilde{t})\tilde{t}d\tilde{t} + \frac{A}{2J}t^2 \tag{5.7}$$

6. RESULTS OF CALCULATIONS

The reduction method was used to calculate the mechanical characteristics in the solution of system of the form (4.7). In order to achieve an accuracy $\varepsilon = 10^{-6}$ in the calculations, it was found to be sufficient to retain 10–12 terms of the expansion. A plastic was chosen as the material of the elastic medium.

The dependence of the stress intensity factor (SIF) on the parameter t^* for different values of the crack opening angle ω when $a_1 = 2a_0$ is shown in the upper part of Fig. 1. Here

$$N_+^* = \frac{N_+}{(a_1 - a_0)\tau_0}, \quad t^* = \frac{tc}{a_1 - a_0}$$

It can be seen that the values of the SIF increase as the crack opening angle increases and the peaks of these values are practically coincident in time.

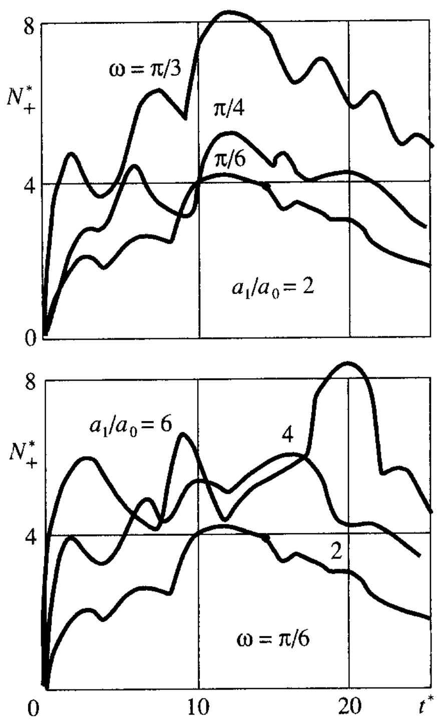

Fig. 1

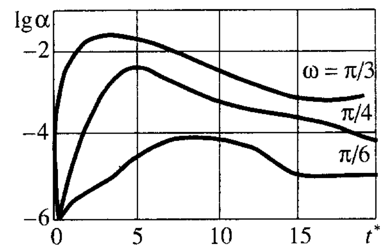

Fig. 2

Numerical calculations showed that the required accuracy is maintained up to a time value 25 and, for long times, the error which is accumulated leads to a sharp deterioration in the convergence of the calculation.

Graphs of the SIF in the case of the crack opening angle $\omega = \pi/6$ for different relative linear dimensions of the crack are shown in the lower part of Fig. 1. Not only is there an increase in the absolute values of the SIF as the linear dimensions increase but, there is also substantial change in the time at which a peak in the value of the SIF occurs: the greater the linear dimension of the crack, the later the time at which the maximum value of the SIF occurs.

The logarithm of the angle of rotation of the inclusion (its linear size $a_1 = 2a_0$) as a function of the dimensionless time for different values of the aperture angle is shown in Fig. 2. The material of the inclusion is aluminium. It is clear that the angle of rotation of the inclusion becomes larger as the opening angle of the defect increases, which is explained by the substantial effect of inertial forces, that depend on the mass of the inclusion, which increases as the dimensions of the defect increase.

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Translated by E.L.S.